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APPROXIMATIONS TO THE DISTRIBUTION FUNCTION OF THE ANDERSON-DARLING TEST STATISTIC

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ABSTRACT. We give a new series representation for the distribution function of the Anderson-Darling test statistic in terms of well known functions. In order to derive fast approximations we tabulate the Anderson-Darling test statistic. As result we present a family of approximations with maximum absolute deviation of 10^{-2} down to 10^{-8}.

1. Introduction

The Anderson-Darling test is one of the most common goodness of fit tests. It was introduced by Anderson and Darling [1] and Darling [3]. In contrast to the somewhat simpler Kolmogoroff-Smirnov test, it is sensitive to deviations in the tails between the empirical and the theoretical distributions. A first table was given by Lewis [6]. Sinclair and Spurr [8] gave an approximation for the whole positive real domain. We give a new representation for the distribution function by evaluating the integrals of Anderson and Darlings representation. In the second part we improve the accuracy of the approximation introduced by Spurr considerably.

2. Computing the Anderson-Darling function

Given a sample of n independent identically distributed random variables T_i with common distribution function $F(t) = P(T_i < t)$, we denote by $T_{(i)}$ its order statistic, that is the permutation of T_i into ascending order. The Kolmogorov-Smirnov tests compare the empirical distribution function $F_n(t)$, which is given by

$$F_n(t) = \begin{cases} 0, & \text{if } t < T_{(1)}, \\ \frac{i}{n}, & \text{for } T_{(i)} \leq t < T_{(i+1)}, \\ 1, & \text{if } T_{(n)} \leq t, \end{cases} \tag{1}$$

to the hypothetisized continuous distribution function $F(t)$. In the case of the Anderson-Darling test, the test statistic is defined as

$$A_n = n \int_{-\infty}^{\infty} \frac{(F_n(t) - F(t))^2}{F(t)\,(1 - F(t))}\, dF(t). \tag{2}$$

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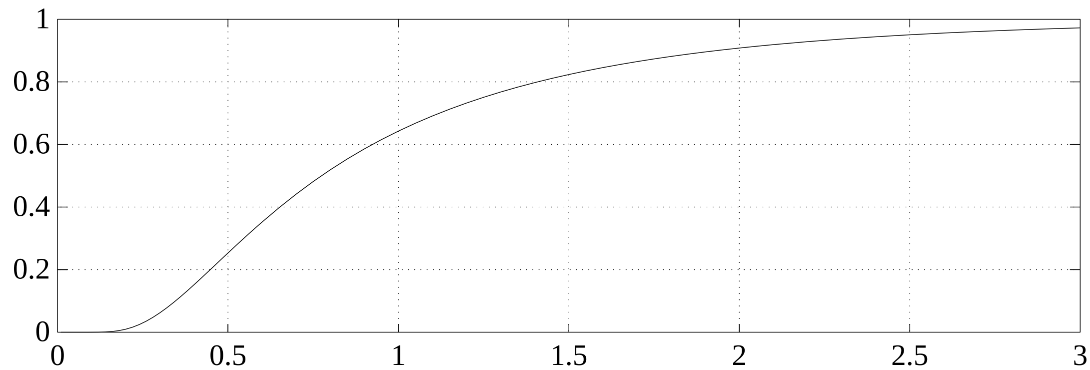

FIGURE 1. The Anderson-Darling function $A(x)$.

With $F_{(i)} = F(T_{(i)})$, one gets by straightforward integration

$$A_n = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\ln F_{(i)} + (2(n-i)+1)\ln(1-F_{(i)}). \tag{3}$$

Numerical simulations from Lewis [6] show a rapid convergence of $\mathrm{P}(A_n < x)$ towards the limiting distribution function $A(x) = \lim_{n\to\infty} \mathrm{P}(A_n < x)$. Anderson and Darling gave a first representation for $A(x)$ (p. 204 in [1]):

$$A(x) = \frac{\sqrt{2\pi}}{x}\sum_{i=0}^{\infty}\binom{-\frac{1}{2}}{i}(4i+1)\exp\left(-\frac{b_i}{x}\right)\int_0^{\infty}\exp\left(\frac{x}{8(w^2+1)} - \frac{b_i w^2}{x}\right)dw,$$
$$b_i = \frac{1}{8}(4i+1)^2\pi^2.$$

From this we obtain, after some manipulations:

$$\begin{aligned} A(x) &= \sum_{i=0}^{\infty}(-1)^i\alpha_i(x), \qquad \alpha_i(x) = \frac{2}{\sqrt{x}}\exp\left(\frac{x}{8} - \frac{\pi^2}{8x}\right)\, a_i(x)\, I(x,b_i), \\ a_i(x) &= \left(1 - \frac{1}{2i}\right)\exp\left(-\frac{(4i-1)\pi^2}{x}\right)a_{i-1}(x), \qquad a_0(x) = 1, \\ b_i &= \frac{1}{8}(4i+1)^2\pi^2 \end{aligned} \tag{4}$$

with

$$\begin{aligned} I(x,b) &= \int_0^{\infty} f(y;x,b)\,dy \qquad x \geq 0, \quad b > 0, \\ f(y;x,b) &= \frac{2}{\sqrt{\pi}}\exp\left(-y^2\left(1 + \frac{x^2}{8b}\cdot\frac{1}{1+xy^2/b}\right)\right). \end{aligned} \tag{5}$$

Figure 1 gives a plot of the Anderson-Darling function for x of moderate size. The calculation of the integrals (5) is the most expensive part of the summation (4). From (5) we get $0 \leq I(x,b) \leq \frac{2}{\sqrt{\pi}}\int_0^{\infty}\exp(-y^2)\,dy = 1$ and the sequence $(I(x,b_i))_i$ is monotonically increasing with limit 1. Furthermore,

$$0 \le \alpha_i(x) \le \alpha_i^*(x) := \frac{2}{\sqrt{x}} \prod_{j=1}^{i} \left(1 - \frac{1}{2j}\right) \exp\left(\frac{x}{8} - \frac{b_i}{x}\right). \tag{6}$$

Therefore, $\alpha_{k+1}^*(x)$ gives an error bound for the k-th partial sum of the alternating series $A(x)$. In Table 1 we show the rapid convergence of series (4).

$x \le$	0.39	0.85	1.94	1.29	2.81	6.43
$A(x) \le$	0.14	0.55	0.90	0.76	0.96	0.999
$A(x) =$		$\alpha_0(x) + \eta$			$\alpha_0(x) - \alpha_1(x) + \eta$	
$\lvert\eta\rvert \le$	7.5e-35	2.1e-16	1.1e-7	1.8e-34	2.2e-16	1.2e-7

TABLE 1. Remainder $\eta = A(x) - \sum_{i=0}^{k}(-1)^i \alpha_i(x)$.

3. Evaluation of $A(x)$ by numerical calculation of the integrals $I(x,b)$

$A(x)$ is approximated by its partial sum s_k with k terms. k is chosen, depending on x, such that the remainder α_{k+1}^* (6) has relative accuracy $\alpha_{k+1}^* < \epsilon s_k$. Here ϵ denotes the relative machine precision (s.p.: ϵ = 1.19e-7, d.p.: ϵ = 2.22e-16, q.p.: ϵ = 1.92e-34). Due to the exponential in (6) one needs only few terms of the series (4).

For numerical integration of $I(x,b)$ we split the improper integrals into two parts:

$$I(x,b) = I_N(x,b) + R_N(x,b), \qquad \text{with}$$

$$I_N(x,b) = \int_0^N f(y;x,b)\,dy \qquad \text{and} \qquad |R_N(x,b)| \le \frac{1}{\sqrt{\pi}N} \cdot \exp(-N^2).$$

We have $|R_N(x,b)| < \epsilon$ for $N \ge 3.75$ (s.p.), 5.81 (d.p.), resp., 8.66 (q.p.). The proper integrals were calculated by using an adaptive Gauss integrator of order 50 and were were checked with the routine DQAGS of the QUADPACK integration package [7].

With this procedure we calculated a large table for the Anderson Darling distribution that is correct to at least 25 digits. These values were used to check and derive further results of this paper. A short extract of this table is given in the appendix in table 5.

$A(x)$ is bounded by α_0^*. Hence, $A(x)$ is effectively zero, viz. $A(x) < u$, if $x \le$ 1.34e-2 (s.p.), 1.72e-3 (d.p.), resp., 1.08e-4 (q.p.). u is the underflow threshold (s.p.: u = 2.93e-39, d.p.: u = 5.56e-309, q.p.: u = 8.40e-4933). The case $A(x) \ge 1 - \epsilon$ is more difficult to handle. From the 25 digit table mentioned above we get $A(x) = 1$ with machine precision for $x \ge 14.57$ (s.p.), 34.25 (d.p.), resp., 80 (q.p.). The value for q.p. is just a rough guess.

4. Representation for the Integrals $I(x,b)$

In order to derive an analytic recpresentation of the Integrals $I(x,b)$ we introduce $I^*(r,s) = I(x,b)$ with $r = x^2/8b$ and $s = b/x$:

$$I^*(r,s) = 2\sqrt{\frac{s}{\pi}} \int_0^\infty \exp\left(-sy^2\left(1 + \frac{r}{1+y^2}\right)\right) dy. \tag{7}$$

Due to

$$\frac{\partial^k I^*(r,s)}{\partial r^k} = 2(-1)^k s^k \sqrt{\frac{s}{\pi}} \int_0^\infty \left(\frac{y^2}{1+y^2}\right)^k \exp\left(-sy^2\left(1+\frac{r}{1+y^2}\right)\right) dy$$

the derivatives are uniformly bound in r :

$$0 < (-1)^k \cdot \frac{\partial^k I^*(r,s)}{\partial r^k} \leq 2s^k \sqrt{\frac{s}{\pi}} \int_0^\infty \exp(-sy^2)\, dy = s^k$$

and, therefore, $I^*(r,s)$ can be expanded in a Taylor series with respect to the variable r :

$$I^*(r,s) = \sum_{k=0}^{\infty} (-1)^k \beta_k(r,s), \qquad \beta_k(r,s) = \frac{1}{k!} I_{k,k}(s) \cdot (rs)^k \tag{8}$$

with

$$I_{k,l}(s) = 2\sqrt{\frac{s}{\pi}} \int_0^\infty \frac{y^{2k}\exp(-sy^2)}{(1+y^2)^l}\, dy. \tag{9}$$

Since $0 < I_{k+1,k+1}(s) < I_{k,k}(s) \leq 1$ series (8) can be majorized by $\exp(rs) = \exp(x/8)$ and, therefore by the results of Lebesgue integration theory, converges on the whole positive real domain. The remainder of the alternating series (8) is restricted by $\beta_{k+1}(r,s)$ with

$$0 < \beta_k(s,r) \leq \beta_k^*(r,s) := \frac{1}{k!}(rs)^k = \frac{1}{k!}\left(\frac{x}{8}\right)^k. \tag{10}$$

The convergence of series (8) is quite fast, too, as can be seen in Table 2.

$x <$	0.1	0.5	1	2	5
η = 1.19e-7	3	4	5	6	8
η = 2.22e-16	6	8	10	11	15
η = 1.92e-34	12	16	18	21	27

TABLE 2. Number of summands k for $|I^*(r,s) - \sum_{l=0}^{k}(-1)^l \beta_l(r,s)| \leq \eta$, $rs = x/8$.

For application we just have to compute the integrals $I_{k,l}(s)$. By partial integration we get the recursion formulas

$$\begin{aligned} I_{k,l}(s) &= I_{k-1,l-1}(s) - I_{k-1,l}(s), \\ I_{k,l}(s) &= \frac{2k-1}{2(l-1)} I_{k-1,l-1}(s) - \frac{s}{l-1} I_{k,l-1}(s). \end{aligned} \tag{11}$$

For $I_{k,k}(s)$ we only need to compute the diagonals $I_{i,i}(s)$ and $I_{i,i+1}(s)$, $i = 0, 1, \ldots, k-1$:

$$\begin{pmatrix} I_{k,k} \\ I_{k,k+1} \end{pmatrix} = M_k \begin{pmatrix} I_{k-1,k-1} \\ I_{k-1,k} \end{pmatrix}, \quad \text{with} \quad M_k = \begin{pmatrix} 1, & -1 \\ -\frac{s}{k}, & \frac{2s+2k-1}{2k} \end{pmatrix}. \tag{12}$$

A simple induction argument yields $I_{k,k}(s) = P_k(s) I_{0,0}(s) + Q_k(s) I_{0,1}(s)$ with P_k and Q_k are polynomials in s of degree less equal k. $I_{0,0}$ and $I_{0,1}$ may be obtained from any integral table (e.g: Gröbner and Hofreiter [5]):

$$I_{0,0} = 1, \qquad I_{0,1} = \sqrt{s\pi}\, e^s \left(1 - \phi(\sqrt{s})\right), \tag{13}$$

where $\phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\, dt$ denotes the error function.

One has to be carefull when using recursion (11) since recursions may enlarge errors in the input data dramatically. An upper bound for the amplification of errors in the input data is given by the largest eigenvalue of M_k. For (11) the eigenvalues are given by $\lambda_{1,2} = \frac{2s+4k-1\pm\sqrt{4s(s+4k-1)+1}}{4k}$. The larger eigenvalue is bounded by $\lambda_m \leq \frac{2s+4k-1}{2k} < \frac{s}{k}+2$ and from 6 we get the bound $s \leq \frac{x}{8} + \ln 2 - \frac{\ln x}{2} - \ln\epsilon \approx 2 - \ln\epsilon$ with ϵ the machine precision. In d.p. the largest eigenvalue ranges from 40 (k = 1) to 2.8 (k = 33). Therefore the worst case error bounds seems to point to a weak instability of the recursion.

On the other hand one can do some numerical simulations in order to study the real behaviour of the recursion. A first step is to study the recursion due to errors in the input data ($I_{0,1}$). These experiments lead to a maximal error amplification for $I_{k,k}$ of less then 100, which is far less than the worst case error bounds propose. A second test is to calculate the Anderson Darling function with high precision and an independent method. For this purpose we calculated the Anderson Darling function in q.p. using numerical integration as described in the last section. Then comparing the q.p. 'exact' values with d.p. function evaluations using the recursion yields about 14 significant digits for arguments x in the range of 0.001 to 20.

5. Computing simultaneously $A(x)$ and its derivative

6. Approximations to the Anderson-Darling function

In the following we give some useful approximations to the Anderson-Darling function (4). We investigate models of the form

$$A_{m,n}(x) = \frac{1}{1 + \exp\left(\sum_{i=-m}^{n} c_i x^{i/2}\right)}. \tag{14}$$

The unknown coefficients c_i, $i = -m, \ldots, n$ were computed by fitting $A_{m,n}(x)$ to $A(x)$ for 1200 geometrically distributed values of x in the range of 0.01 to 30, viz., $x_i = 10^{-2}\exp(8i/N)$, $i = 1, \ldots, N = 1200$. Function values with at least 25 significant digits were computed in q.p. to allow approximations of considerable accuracy.

The optimization was done by minimizing the squared absolute deviation $\epsilon^{(2)}_{m,n}$. However, for practical purposes the maximal absolute deviation $\epsilon^{(\infty)}_{m,n}$ is of superior interest:

$$\epsilon^{(\infty)}_{m,n} = \max_{i=1,\ldots,N} \left|A_{m,n}(x_i) - A(x_i)\right|, \qquad \epsilon^{(2)}_{m,n} = \sqrt{\frac{1}{N}\sum_{i=1}^{N} \left(A_{m,n}(x_i) - A(x_i)\right)^2}. \tag{15}$$

We used a variant of the damped Gauss-Newton algorithm for overdetermined systems of equations (Deuflhard [4]). Let $\mathbb{J} = \left(\partial A_{m,n}(x_i)/\partial c_j\right)_{i,j}$ be the Jacobian of $A_{m,n}$, $\mathbb{J}^+$ its Moore-Penrose inverse (Campbell and Meyer [2]), and $\mathbf{r} = (A_{m,n}(x_i) - A(x_i))_i$ the residuals of the approximation then the norm of the gradient $g_{m,n}$ with respect to the parameters c_i,

$i = -m, \ldots, n$ of $\epsilon^{(2)}_{m,n}$ is given by

$$g_{m,n} = \max_{-m \le j \le n} \left| \frac{\partial}{\partial c_j} \epsilon^{(2)}_{m,n} \right| = \left\| \frac{1}{N \epsilon^{(2)}_{m,n}} \mathbb{J}^t \cdot \mathbf{r} \right\|_\infty , \tag{16}$$

and the norm of the Newton correction $n_{m,n}$ is given by

$$n_{m,n} = \| \mathbb{J}^+ \mathbf{r} \|_\infty . \tag{17}$$

Both the gradient and the Newton correction should be zero for any approximation of type (14). In order to achieve asymptotic accuracy, that means $\lim_{x \searrow 0} A_{m,n}(x) = 0$ and $\lim_{x \nearrow \infty} A_{m,n}(x) = 1$, one has to ensure $c_{-m} > 0$ and $c_n < 0$.

model	m	n	c_{-i}	c_i
1	2	2	$c_{-1} =$ 0.0	$c_0 =$ 0.0
			$c_{-2} =$ 0.9070430076	$c_1 =$ 0.0
				$c_2 =$ -1.4589001606
2	2	2	$c_{-1} =$ 0.0	$c_0 =$ -0.4564729970
			$c_{-2} =$ 1.0645978662	$c_1 =$ 0.0
				$c_2 =$ -1.1877352471
3	3	2	$c_{-1} =$ 2.018	$c_0 =$ -1.784
			$c_{-2} =$ -0.03287	$c_1 =$ 0.0
			$c_{-3} =$ 0.2029	$c_2 =$ -0.9936
4	3	2	$c_{-1} =$ 2.7288915985	$c_0 =$ -2.1752775349
			$c_{-2} =$ -0.4796446706	$c_1 =$ 0.0
			$c_{-3} =$ 0.2952587385	$c_2 =$ -0.9559826675
5	6	6	$c_{-1} =$ -21.709875729665958	$c_0 =$ 21.963952580957401
			$c_{-2} =$ 17.02784409390403	$c_1 =$ -16.0241799316907
			$c_{-3} =$ -8.451271467502671	$c_2 =$ 5.170083836170379
			$c_{-4} =$ 2.838905632872688	$c_3 =$ -0.639567071529326
			$c_{-5} =$ -0.522642367619939	$c_4 =$ -0.426622798284516
			$c_{-6} =$ 0.04012023171262011	$c_5 =$ 0.163706780494995
				$c_6 =$ -0.01770306898191751
6	9	8	$c_{-1} =$ 600.95571175660280345225	$c_0 =$ -659.26172666173619823237
			$c_{-2} =$ -432.89391432660725284049	$c_1 =$ 571.16951756610235048647
			$c_{-3} =$ 247.28083894485872790373	$c_2 =$ -389.28185707281383978447
			$c_{-4} =$ -109.86875110264501189780	$c_3 =$ 202.11885387348583955119
			$c_{-5} =$ 37.02189383169910854208	$c_4 =$ -78.04432695135807125505
			$c_{-6} =$ -9.03665963173656979822	$c_5 =$ 21.41871180140172046472
			$c_{-7} =$ 1.49139645725983514093	$c_6 =$ -3.92272600670526755828
			$c_{-8} =$ -0.14776885542449894026	$c_7 =$ 0.42792196483150181467
			$c_{-9} =$ 0.00660542586887867715883	$c_8 =$ -0.02096837085089079908540

TABLE 3. Coefficients for approximations of the Anderson-Darling function $A(x)$.

model	$\epsilon^{(\infty)}_{m,n}$	$\epsilon^{(2)}_{m,n}$	$g_{m,n}$	$n_{m,n}$
1	1.40e-2	5.45e-3	1.45e-10	2.65e-11
2	2.47e-3	8.97e-4	1.66e-9	3.94e-11
3	1.12e-3	3.72e-4	4.21e-2	0.71
4	1.48e-4	6.11e-5	1.36e-8	4.61e-11
5	5.89e-7	2.32e-7	6.42e-10	1.08e-11
6	2.49e-8	8.24e-9	7.91e-12	4.14e-6

TABLE 4. Error-coefficients for approximations of the Anderson-Darling function $A(x)$.

In Table 3 we give the coefficients for 6 different approximations to the Anderson Darling function and Table 4 we give the corresponding approximation levels $\epsilon^{(\infty)}_{m,n}$ and $\epsilon^{(2)}_{m,n}$. Furthermor we give norm of the gradient $g_{m,n}$ as well as the norm of the Newton correction $n_{m,n}$ of the optimization objective $\epsilon^{(2)}_{m,n}$.

Models 1 and 2 give very fast approximations which use only 3 multiplications, one root, and one exponentiation. The accuracy is high enough for most practical purposes. Model 3, which is due to Sinclair and Spurr [8] was improved considerably – the result is given in model 4. Models 5 and 6 are approximations with enhanced precision, where the computation of equations (4,5) is quite expensive.

7. CONCLUSION

In this paper we have given a new representation for the Anderson-Darling function and we have obtained some practical approximations for the Anderson-Darling function. All approximations are uniformly bound by $\epsilon^{(\infty)}$ given in Table 4. There exist quite accurate approximations which need only few multiplications, one root, and one exponentiation. In d.p. model 5 gives the best approximation. The approximation of model 6 should, due to roundoff errors, used for q.p. calculations only.

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Appendix A. Table of the Anderson-Darling function

To compute the approximations of the Anderson-Darling function, $A(x)$ was tabulated with an accuracy of about 25 digits. In Table 5 we give a shortened version[1]:

Table 5. The Anderson-Darling function $A(x)$

x	$\lim_{n\to\infty} P(A_n < x)$	x	$\lim_{n\to\infty} P(A_n < x)$	x	$\lim_{n\to\infty} P(A_n < x)$
0.00	0.0000000000	0.60	0.3520043643	3.00	0.9726352117
0.01	0.5280032130E-52	0.65	0.3979772117	3.20	0.9783148210
0.02	0.2302104730E-25	0.70	0.4411767961	3.40	0.9827780093
0.04	0.4049527272E-12	0.75	0.4815017531	3.60	0.9862964958
0.06	0.9667522794E-08	0.80	0.5189720470	3.80	0.9890778732
0.08	0.1433284070E-05	0.85	0.5536823836	4.00	0.9912818131
0.10	0.2807810513E-04	0.90	0.5857708152	4.50	0.9950099735
0.12	0.2007999640E-03	0.95	0.6153979343	5.00	0.9971255787
0.14	0.8093318094E-03	1.00	0.6427333268	5.50	0.9983358392
0.16	0.2282938353E-02	1.10	0.6912037862	6.00	0.9990325481
0.18	0.5081160191E-02	1.20	0.7324651158	6.50	0.9994356458
0.20	0.9587452750E-02	1.30	0.7676522535	7.00	0.9996698332
0.22	0.1605076086E-01	1.40	0.7977343134	7.50	0.9998063597
0.24	0.2457480960E-01	1.50	0.8235246272	8.00	0.9998861858
0.26	0.3513705419E-01	1.60	0.8457003095	8.50	0.9999329776
0.28	0.4761981515E-01	1.70	0.8648234033	9.00	0.9999604660
0.30	0.6184236394E-01	1.80	0.8813604210	9.50	0.9999766455
0.32	0.7758846955E-01	1.90	0.8956992386	10.00	0.9999861850
0.34	0.9462763170E-01	2.00	0.9081632251	11.00	0.9999951489
0.36	0.1127301752	2.10	0.9190228580	12.00	0.9999982897
0.38	0.1316771904	2.20	0.9285051875	13.00	0.9999993950
0.40	0.1512664993	2.30	0.9368015178	14.00	0.9999997854
0.42	0.1713157335	2.40	0.9440736357	15.00	0.9999999237
0.44	0.1916634169	2.50	0.9504588656	16.00	0.9999999728
0.46	0.2121687356	2.60	0.9560741815	17.00	0.9999999903
0.48	0.2327104965	2.70	0.9610195604	18.00	0.9999999965
0.50	0.2531856265	2.80	0.9653807281	19.00	0.9999999988
0.55	0.3035487164	2.90	0.9692314140	20.00	0.9999999996

Appendix B. Implementation

In the following we give implementations for exact and approximative computaion of the Anderson-Darling function. All of the functions ad_fcn, $adtfcn$, $adqfcn$, $adrfcn$ compute the series series (4). These routined differ by calculating the integrals $I(x,b)$ of (5). ad_fcn is a synonym for $adqfcn$, $adtfcn$ uses TEGRAL, $adqfcn$ uses QUADPACK, and $adrfcn$ uses the recursion (11). ad_apr, $ad1apr$ … $ad6apr$ denote the impementagion of the Approximations

[1]The full table as well as the underlying Fortran sources can be obtained from Mössner.

of Table 3. *ad_apr* again is a general purpose routine which uses, depending on the machine precision, the best version.

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